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Properties of 2×2 *h*-Deformed Quantum (Super)Matrices

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We investigate the h-deformed quantum (super)group of 2×2 matrices and use a kind of contraction procedure to prove that the n-th power of this deformed quantum (super)matrix is quantum (super)matrix with the deformation parameter nh.

KEY WORDS: h-deformed; R-matrix; quantum group; quantum matrix.

There are two distinct quantizations for general Lie group. One is the well-known Drinfeld–Jimbo q-deformed quantum group and the other is the so-called Jordanian h-deformed one. For group $GL(2)$ there exist only such two deformations (up to isomorphism) with a central quantum determinant: $GL_q(2)$ and $GL_h(2)$ (Kupershmidt, 1992). One can obtain these quantum groups by deforming the coordinates of a linear plane to be noncommutative objects. In detail, the deformed quantum groups act on the q-plane, with the relation $xy = qyx$ and the h-plane, with the relation $xy - yx = hy^2$, respectively. The q-deformed quantum group has been intensively studied (Drinfeld, 1986; Faddeev *et al.*, 1989) because it is closely related with the solution of Yang–Baxter equation and the theory of braids. In Lukierski *et al.* (1995), it is found that the deformation parameter of h-deformed quantum group is naturally dimensional (like the known k-deformation). This is quite intriguing for possible physical applications. Recently geometrical structure of the h-plane has been thoroughly discussed (Cho *et al.*, 1998) and quantum mechanics based on the plane was constructed (Cho, 1999). These works are helpful to understand h-deformed quantum group.

In Corrigan *et al.* (1990) and Vokos *et al.* (1990), q-deformed quantum group $GL_q(2)$ was investigated and it was shown that the n-th power of a q-deformed

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quantum matrix corresponds to the n-th power of the deformation parameter q. Also the same property of q-deformed quantum supergroup $GL_q(1|1)$ was discussed in Schwenk *et al.* (1990). And in Kupershmidt (1992), it was pointed out that the n-th power of a h-deformed quantum matrix is also quantum matrix with the deformation parameter nh. In Aghamohammadi *et al.* (1995), it was found that the h-deformed quantum group can be obtained from the q-deformed quantum Lie group through a singular limit $q \to 1$ of a linear transformation. This method is called as the contraction procedure.

In this paper we use the contraction method to obtain the same result pointed out in Kupershmidt (1992). We also extend the discussion to deformed quantum supergroup and show that the n-th power of a h-deformed quantum supermatrix is also quantum supermatrix with the deformation parameter nh. Not only do we use the sigular linear transformation between the two kinds of quantum (super) groups to obtain the results, but also we use the similarity transformation between the corresponding R-matrices for different quantum groups to obtain them.

To begin with we define the q-deformed quantum matrix as

$$M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}. \tag{1}$$

Throughout this paper we denote q-deformed objects by primed quantities. The unprimed ones represent tansformed deformed objects. The Manin quantum plane satisfying the commutation relation $xy = qyx$ can be considered covariant with respect to the action of the quantum group $GL_q(2)$. $M' \in GL_q(2)$ means the following commutation relations are fulfiled

$$\begin{aligned} &a'c' = qc'a', \quad b'd' = qd'b', \quad [a', d'] = (q - q^{-1})b'c', \\ &a'b' = qb'a', \quad c'd' = qd'c', \quad c'b' = b'c'. \end{aligned} \tag{2}$$

Here [,] stands for the commutator. The quantum determinant D', which is defined as

$$D' = \det_q M' \equiv a'd' - qc'b', \tag{3}$$

is central. One can obtain h-deformed quantum group $GL_h(2)$ with the following transformation:

$$M' = gMg^{-1}, \tag{4}$$

where

$$g = \begin{pmatrix} 1 & \frac{h}{q-1} \\ 0 & 1 \end{pmatrix}. \tag{5}$$

A simple calculation shows that the q-deformed matrix elements and the corresponding tansformed ones fulfil the following relations:

$$\begin{aligned} a' &= a + \frac{h}{q-1}c, \quad & b' &= b + \frac{h}{q-1}(d-a) - \frac{h^2}{(q-1)^2}c, \\ c' &= c, & d' &= d - \frac{h}{q-1}c. \end{aligned} \tag{6}$$

Substituting the above relations into the q-commutation relations (2) and taking the limit of $q \to 1$, one can find that the entries of the transformed quantum matrix M satisfy the following commutation relations of the $GL_h(2)$:

$$\begin{aligned} &[a, c] = hc^2, \quad [d, b] = h(D_h - d^2), \quad [a, d] = hdc - hac, \\ &[d, c] = hc^2, \quad [b, c] = hac + hcd, \quad [b, a] = h(a^2 - D_h). \end{aligned} \tag{7}$$

Here the corresponding h-deformed determinant is

$$D_h = \det_h M \equiv ad - cb - hcd, \tag{8}$$

which is also central. The h-deformed quantum plane satisfying the relation $xy = yx + hy^2$ can also be considered covariant with respect to the action of the above quantum group $GL_h(2)$.

For n-th power of M', from (4), we obtain

$$\begin{aligned} M'^n &= (gMg^{-1})^n \\ &= gM^n g^{-1}. \end{aligned} \tag{9}$$

If we denote M'^n, M^n as $\left(\begin{smallmatrix} a'_n & b'_n \\ c'_n & d'_n \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$, respectively, the relations between them can be obtained in a similar way to get Eq. (6). They are given by

$$\begin{aligned} a'_n &= a_n + \frac{h}{q-1}c_n, \quad & b'_n &= b_n + \frac{h}{q-1}(d_n - a_n) - \frac{h^2}{(q-1)^2}c_n, \\ c'_n &= c_n, & d'_n &= d_n - \frac{h}{q-1}c_n. \end{aligned} \tag{10}$$

But because M'^n belongs to $GL_{q^n}(2)$ (Corrigan *et al.*, 1990; Vokos *et al.*, 1990), the matrix elements of M'^n will satisfy the commutation relations (2) only with the deformed parameter q replaced by q^n. So we can have the following relations:

$$\begin{aligned} &a'_n c'_n = q^n c'_n a'_n, \quad b'_n d'_n = q^n d'_n b'_n, \quad [a'_n, d'_n] = (q^n - q^{-n}) b'_n c'_n, \\ &a'_n b'_n = q^n b'_n a'_n, \quad c'_n d'_n = q^n d'_n c'_n, \quad c'_n b'_n = b'_n c'_n. \end{aligned} \tag{11}$$

Following the same procedure as we obtain relations (7) and with the limit identity

$$\lim_{q\to 1}\frac{q^n-1}{q-1}=n, \tag{12}$$

we can have the following commutation relations of the $GL_{nh}(2)$ given by

$$\begin{aligned} &[a_n, c_n] = nhc_n^2, && [d_n, b_n] = nh\big(D_{nh} - d_n^2\big),\\ &[a_n, d_n] = nhd_nc_n - nha_nc_n, && [d_n, c_n] = nhc_n^2,\\ &[b_n, c_n] = nha_nc_n + nhc_nd_n, && [b_n, a_n] = nh\left(a_n^2 - D_{nh}\right), \end{aligned} \tag{13}$$

where

$$D_{nh} = \det_h M^n \equiv a_nd_n - c_nb_n - nhc_nd_n, \tag{14}$$

which mean that M^n does belong to the h-deformed quantum group $GL_{nh}(2)$.

In fact, by using the R-matrix for q-deformation (Corrigan *et al.*, 1990; Vokos *et al.*, 1990):

$$R' = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & q & 1-q^2 & 0\\ 0 & 0 & q & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{15}$$

one can obtain the commutation relations (2) between the elements of q-deformed matrix M' through the following equation:

$$R'M_1'M_2' = M_2'M_1'R', \tag{16}$$

where M_1', M_2' are the graded matrices with the following matrix elements respectively

$$\begin{aligned} (M_1')^{ab}_{cd} &= (M'\otimes I)^{ab}_{cd} = M_c'^{a}\delta^b_d,\\ (M_2')^{ab}_{cd} &= (I\otimes M')^{ab}_{cd} = M_d'^{b}\delta^a_c. \end{aligned} \tag{17}$$

And it is clear that the transformation between the deformed quantum groups $M' = gMg^{-1}$ leads to the similarity transformation between the corresponding R-matrices for different quantum groups:

$$R = (g\otimes g)^{-1}\,R'(g\otimes g). \tag{18}$$

Here product $\otimes$ acts in the same manner as M_1' and M_2' expressions (17). Through such transformation the R-matrix for the h-deformation can be obtained as the following:

$$R_h \equiv \lim_{q\to 1} R = \begin{pmatrix} 1 & -h & h & h^2 \\ 0 & 1 & 0 & -h \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

Applying Eqs. (4) and (18), one can obtain the following equation in the limit of $q \to 1$:

$$R_h M_1 M_2 = M_2 M_1 R_h. \tag{20}$$

So the h-commutation relations (7) can be derived from the above equation again. As a common sense, the R-matrix for q-deformation in the quantum group $GL_{q^n}(2)$ is given by

$$R_n' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^n & 1 - q^{2n} & 0 \\ 0 & 0 & q^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{21}$$

Because of the transformation (9), we also have

$$R_n = (g \otimes g)^{-1} R_n' (g \otimes g). \tag{22}$$

So with the limit identity (12), we can obtain the R-matrix for h-deformation in the quantum group $GL_{nh}(2)$ following by

$$R_{nh} \equiv \lim_{q\to 1} R_n = \begin{pmatrix} 1 & -nh & nh & (nh)^2 \\ 0 & 1 & 0 & -nh \\ 0 & 0 & 1 & nh \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

Also from Eqs. (9) and (22), we derive the following equation in the limit of $q \to 1$:

$$R_{nh} M_1^n M_2^n = M_2^n M_1^n R_{nh} \tag{24}$$

and the commutation relations Eq. (13) of the quantum group $GL_{nh}(2)$ are obtained through this equation and it shows that M^n definitely belongs to $GL_{nh}(2)$.

Next step, we will extend our discussion to quantum supergroup. We denote supermatrix quantity as $\tilde{Q}$ with Q representing the quantity for the common deformed group. Returning to the Manin quantum superplane satisfying the relation $x\theta = q\theta x$, $\theta^2 = 0$, there exists the R-matrix for the quantum supergroup

$GL_q(1 \mid 1)$ acting on the superplane (Schmidke *et al.*, 1990), where

$$\tilde{R}' = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^1 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}. \tag{25}$$

The h-deformed quantum superplane satisfying the relation $x\theta = \theta x + hx^2, \theta^2 = -h\theta x$, can also be obtained through the singular transformation in the limit $q \to 1$. Conventionally, here we take the transfer matrix g as $\left(\begin{smallmatrix} 1 & 0 \\ \frac{h}{q-1} & 1 \end{smallmatrix}\right)$. Observing that for h-deformed quantum supergroup the deformation parameter must be a dual odd (Grassmann) number satisfying $h^2 = 0$ (Dabrowski and Parashar, 1996), one can obtain the corresponding R-matrix for the quantum supergroup $GL_h(1 \mid 1)$ acting on the h-deformed quantum superplane through the similarity transformation in Eq. (18):

$$\tilde{R}_h \equiv \lim_{q \to 1} \tilde{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -h & 1 & 0 & 0 \\ h & 0 & 1 & 0 \\ 0 & h & h & 1 \end{pmatrix}. \tag{26}$$

Here, the only difference is the grading product $\otimes$. For quantum supergroup, the product acts in the following manner:

$$\begin{aligned} (\tilde{M}'_1)^{ab}_{cd} &= \left(\tilde{M}' \otimes I\right)^{ab}_{cd} = (-1)^{c(b+d)} \tilde{M}'^a_c \delta^b_d, \\ (\tilde{M}'_2)^{ab}_{cd} &= \left(I \otimes \tilde{M}'\right)^{ab}_{cd} = (-1)^{a(b+d)} \tilde{M}'^b_d \delta^a_c. \end{aligned} \tag{27}$$

From the same relation as Eq. (20), one can obtain the commutation relations of quantum supergroup $GL_h(1 \mid 1)$, which is given by

$$\begin{aligned} &ab = ba, && ac = ca + h(a^2 + cb - da) \\ &bd = db, && cd = dc + h(a^2 - cb - da) \\ &b^2 = 0, && c^2 = h(dc - ca) \\ &bc = -cb - h(ba - db), && ad = da + h(ba - db). \end{aligned} \tag{28}$$

The n-th power of supermatrix $\tilde{M}'$ belongs to quantum supergroup $GL_{q^n}(1 \mid 1)$ and R-matrix for $GL_{q^n}(1 \mid 1)$ is given by

$$\tilde{R}'_n = \begin{pmatrix} q^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^n - q^{-n} & 1 & 0 \\ 0 & 0 & 0 & q^{-n} \end{pmatrix}. \tag{29}$$

So through the transformation like Eq. (22) and with the limit of $q \to 1$, we can have the R-matrix for quantum supergroup $GL_{nh}(1 \mid 1)$ as following:

$$\tilde{R}_{nh} \equiv \lim_{q\to 1} \tilde{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -nh & 1 & 0 & 0 \\ nh & 0 & 1 & 0 \\ 0 & nh & nh & 1 \end{pmatrix}. \tag{30}$$

Also following the same procedure as we obtain Eq. (24), we have the relation for the n-th power of quantum supermatrix $\tilde{M}^n$:

$$\tilde{R}_{nh}\tilde{M}_1^n\tilde{M}_2^n = \tilde{M}_2^n\tilde{M}_1^n\tilde{R}_{nh}, \tag{31}$$

which shows that the entries of the transformed quantum matrix $\tilde{M}^n$, fulfil the commutation relations of the quantum supergroup $GL_{nh}(1 \mid 1)$ as following:

$$\begin{aligned} &ab = ba, && ac = ca + nh(a^2 + cb - da) \\ &bd = db, && cd = dc + nh(d^2 - cb - da) \\ &b^2 = 0, && c^2 = nh(dc - ca) \\ &bc = -cb - nh(ba - db), && ad = da + nh(ba - db). \end{aligned} \tag{32}$$

Hence, we conclude that $\tilde{M}^n$ belongs to $GL_{nh}(1 \mid 1)$.

In summary, we show that the n-th power of h-deformed quantum matrix is quantum matrix with the deformation parameter nh and the same is also true for the n-th power of h-deformed quantum supermatrix.

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